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World Institute for Development Economics Research

Working Paper No. 2012/62

On a Distance Function-Based Inequality Measure in the Spirit of the Bonferroni and Gini Indices

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June 2012

Abstract

A natural way of viewing an inequality or a poverty measure is in terms of the vector distance between an actual (empirical) distribution of incomes and some appropriately normative distribution (reflecting a perfectly equal distribution of incomes, or a distribution with the smallest mean that is compatible with a complete absence of poverty). Real analysis offers a number of distance functions to choose from. In this paper, the employment of what in the literature is known as the Canberra distance function leads to an inequality measure in the tradition of the Bonferroni and Gini indices of inequality. The paper discusses some properties of the measure, and presents a graphical representation of inequality which shares commonalities with the well known Lorenz curve depiction of distributional inequality.

Keywords: Gini coefficient, Bonferroni index, Canberra distance function, poverty, inequality

JEL classification: D63, D31, I32

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This study has been prepared within the UNU-WIDER project 'New Directions in Development Economics'.

UNU-WIDER gratefully acknowledges the financial contributions to the research programme by the governments of Denmark (Royal Ministry of Foreign Affairs), Finland (Ministry for Foreign Affairs), Sweden (Swedish International Development Cooperation Agency—Sida) and the United Kingdom (Department for International Development—DFID).

ISSN 1798-7237 ISBN 978-92-9230-525-3

Acknowledgements

I am indebted to R. Dharumaperumal and A. Arivazhagan for help with the graphics.

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Typescript prepared by Lorraine Telfer-Taivainen at UNU-WIDER

1 Introduction

In this paper, I advance a measure of inequality which is very similar to the Bonferroni (1930) index, and also shares commonalities with the well-known Gini (1912) coefficient of inequality, and an attempt is made in the paper to flag the relevant links. The measure is derived, very simply, as a distance function, and since the specific distance function employed in the cause is the so-called Canberra function (see Lance and Williams 1967) the resulting index is called the 'Canberra inequality measure'. Some features of the measure are discussed, with specific reference to the properties of decomposability and transfer-sensitivity. Also discussed is a graphical representation of inequality, analogous to the Lorenz and Bonferroni curves, which is here called the Canberra curve, from which the Canberra measure can be derived, just as the Gini coefficient can be derived from the Lorenz curve and the Bonferroni index from the Bonferroni curve (on which, see Barcena and Imedio 2000).

At least four important contributions to reviving interest in, to interpreting, to characterizing, and to analyzing the properties of, the Bonferroni inequality measure are the papers by Barcena and Imedio (2000), Giorgi and Crescenzi (2001), Chakravarty (2007), and Imedio-Olmedo, Barcena-Martin and Parrado-Gallardo (2011). Also of relevance is a brief note by the present author (Subramanian 1989) which advanced what was called a 'simple transfer-sensitive index of inequality', in complete ignorance of the fact that the index in question was Bonferroni's!

2 Notation

An ordered income *n*-vector is a list $\mathbf{x}$ of n non-negative incomes $(x_1,...,x_i,...,x_n)$ arranged in non-decreasing sequence (so that $0 \le x_i \le x_{i+1}, i = 1,...,n-1$), where x_i is the income of the *i*th poorest person in a community of *n* individuals, *n* being a member of the set of positive integers $\mathcal{N}$, with person 1 being the poorest individual and person *n* the richest. The set of all *n*-vectors is $\mathbf{X}_n$ and the set of all income distributions is represented by $\mathbf{X} \equiv \cup_{n\in\mathcal{N}} \mathbf{X}_n$. For every $\mathbf{x} \in \mathbf{X}$, the set of individuals whose incomes are represented in $\mathbf{x}$ is designated by $N(\mathbf{x})$, the dimensionality of $\mathbf{x}$ by $n(\mathbf{x})$, and the mean of $\mathbf{x}$ by $\mu(\mathbf{x}) \equiv (1/n(\mathbf{x})) \sum_{i\in N(\mathbf{x})} x_i$. If $\mathcal{R}$ is the set of reals, then an inequality measure is a mapping $I : \mathbf{X} \to \mathcal{R}$ such that, for every $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x})$ is a unique real number that indicates the amount of inequality associated with the vector $\mathbf{x}$. Given any $\mathbf{x} \in \mathbf{X}$, we shall let $\mu_i(\mathbf{x}) \equiv (1/i)\sum_{j=1}^{i} x_j$ stand for the average income of those with incomes not exceeding the *i*th poorest person's income; in the interests of convenience, if not correctness in the use of language, we shall also refer to $\mu_i(\mathbf{x})$ as 'person *i*'s mean income', as in Subramanian (1989). (Notice that $\mu_1(\mathbf{x}) \equiv x_1$ and $\mu_n(\mathbf{x}) \equiv \mu(\mathbf{x})$.) For any $\mathbf{x} \in \mathbf{X}$, we shall define two corresponding distinguished vectors (of the same dimensionality as $\mathbf{x}$), given, respectively by, $\boldsymbol{\mu}_{\mathbf{x}} \equiv (\mu(\mathbf{x}),...,\mu(\mathbf{x}))$ and

$\hat{\boldsymbol{\mu}}_{\mathbf{x}} \equiv (\mu_1(\mathbf{x}),...,\mu_n(\mathbf{x}))$. Where there is no risk of ambiguity, we shall also write μ for $\mu(\mathbf{x})$, μ_i for $\mu_i(\mathbf{x})$, n for $n(\mathbf{x})$, and so on.

3 Poverty and inequality measures as distance functions

Given any three vectors **a**, **b** and **c** in *n*-dimensional real space, the *distance* between the vectors **a** and **b**, represented as $\delta(\mathbf{a},\mathbf{b})$, is a metric which satisfies the properties of non-negativity [namely, $\delta(\mathbf{a},\mathbf{b}) \geq 0$], identity [namely, $\delta(\mathbf{a},\mathbf{a}) = 0$], symmetry [namely, $\delta(\mathbf{a},\mathbf{b}) = \delta(\mathbf{b},\mathbf{a})$], and triangle inequality [namely $\delta(\mathbf{a},\mathbf{b}) + \delta(\mathbf{b},\mathbf{c}) \geq \delta(\mathbf{c},\mathbf{a})$]. Measures of poverty and inequality are essentially measures of distance—between empirical vectors of income and certain idealized vectors, such as vectors of 'no poverty', in the case of poverty measurement, and vectors of equal incomes, in the case of inequality measurement. Given any ordered *n*-vector of incomes $\mathbf{x} = (x_1,...,x_i,...,x_n)$ and the corresponding equally distributed vector of incomes $\boldsymbol{\mu}_{\mathbf{x}} \equiv (\mu(\mathbf{x}),...,\mu(\mathbf{x}))$, one can see that the distance between **x** and $\boldsymbol{\mu}_{\mathbf{x}}$, averaged over the number of individuals in the society $n(\mathbf{x})$, could legitimately be interpreted as a measure of inequality. For any two vectors **x** and **y** in *n*-dimensional real space, the *Euclidean distance* between the vectors is given by: $\delta_E(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{n(\mathbf{x})}(x_i - y_i)^2\right]^{1/2}$. Note now that the well-known inequality measure yielded by the Standard Deviation (S) of incomes ($= (1/n(\mathbf{x}))^{1/2}\left[\sum_{i=1}^{n(\mathbf{x})}(x_i - \mu(\mathbf{x}))^2\right]^{1/2}$) is simply proportional to the Euclidean distance between the income vector **x** one is confronted by and its corresponding $\boldsymbol{\mu}_{\mathbf{x}}$ vector, viz. $S(\mathbf{x}) = (1/\sqrt{n(\mathbf{x})})\delta_E(\mathbf{x},\boldsymbol{\mu}_{\mathbf{x}})$.

Real analysis offers a number of distance functions to choose from—Wilson and Martinez (1997) provides a particularly useful review—and some of these have been employed in the economics measurement literature: Subramanian (2009), for instance, suggests a certain close correspondence between the well-known Foster-Greer-Thorbecke (Foster et al. 1984) family of poverty measures and the class of *Minkowski* distance functions. Of relevance for the purposes of the present paper is the so-called 'Canberra distance function', due to Lance and Williams (1967) (which has also been employed to derive a parametrized family of poverty indices in Subramanian 2009). Given any two vectors **a** and **b** in *n*-Euclidean space, the Canberra distance between the two vectors is defined as:

$$(1)\quad \delta_C(\mathbf{a},\mathbf{b}) = \sum_{i=1}^{N}\left|\frac{a_i - b_i}{a_i + b_i}\right|.$$

It is the distance function featured above which will be employed, in what follows, to derive a variant—the Canberra index—of the Bonferroni inequality measure.

4 The Canberra inequality measure

For any (ordered) income vector $\mathbf{x} \in \mathbf{X}$, and the corresponding vectors $\boldsymbol{\mu}_{\mathbf{x}}$ and $\hat{\boldsymbol{\mu}}_{\mathbf{x}}$ defined in Section 2, we now define the Canberra Inequality Measure C as the Canberra distance between the vectors $\boldsymbol{\mu}_{\mathbf{x}}$ and $\hat{\boldsymbol{\mu}}_{\mathbf{x}}$, averaged across the n individuals constituting the society under review. For all $\mathbf{x} \in \mathbf{X}$:

$$(2)\ C(\mathbf{x}) = (1/n(\mathbf{x}))\delta_C(\boldsymbol{\mu}_{\mathbf{x}}, \hat{\boldsymbol{\mu}}_{\mathbf{x}}) = (1/n(\mathbf{x}))\sum_{i=1}^{n(\mathbf{x})}\left[\frac{\mu(\mathbf{x}) - \mu_i(\mathbf{x})}{\mu(\mathbf{x}) + \mu_i(\mathbf{x})}\right].$$

The (relative) Bonferroni Index—see, for example, Chakravarty (2007)—is given by: for all $\mathbf{x} \in \mathbf{X}$,

$$(3)\ B(\mathbf{x}) = (1/n(\mathbf{x}))\sum_{i=1}^{n(\mathbf{x})}\left[\frac{\mu(\mathbf{x}) - \mu_i(\mathbf{x})}{\mu(\mathbf{x})}\right].$$

As one can see from expressions (2) and (3), the Canberra index is a close relative of the Bonferroni Index, with the difference reflected in the additional term μ_i in the denominator in the square bracket on the right hand side of equation (2). One way of writing the Gini coefficient of inequality—see, for example, Sen (1973)—is the following: for all $\mathbf{x} \in \mathbf{X}$,

$$(4)\ G(\mathbf{x}) = \frac{n(\mathbf{x}) + 1}{n(\mathbf{x})} - \left(\frac{2}{n^2(\mathbf{x})\mu(\mathbf{x})}\right)\sum_{i=1}^{n(\mathbf{x})}(n(\mathbf{x}) + 1 - i)x_i\ .$$

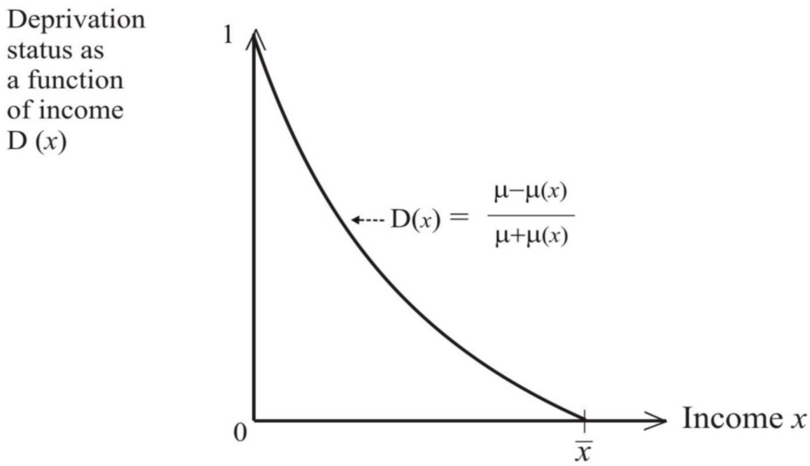
To see the link between the Canberra and Gini measures, it can be verified from equation (2) that, for all $\mathbf{x} \in \mathbf{X}$,

$$(5)\ C(\mathbf{x}) = (1/n(\mathbf{x}))\left[\frac{\mu(\mathbf{x}) - x_1}{\mu(\mathbf{x}) + x_1} + \ldots + \frac{i\mu(\mathbf{x}) - (x_1 + \ldots + x_i)}{i\mu(\mathbf{x}) + (x_1 + \ldots + x_i)} + \ldots + \frac{n(\mathbf{x})\mu(\mathbf{x}) - (x_1 + \ldots + x_n)}{n(\mathbf{x})\mu(\mathbf{x}) + (x_1 + \ldots + x_n)}\right].$$

It may be noted from expression (5) that the income level x_i is repeated $(n + 1 - i)$ times over, for every $i = 1,\ldots,n$: this corresponds exactly with the Borda rank-order weighting scheme that is a distinctive feature of the Gini coefficient; and the 'equity-consciousnes' of these indices is captured precisely in a system of diminishing weights, given by the income levels' respective rank-orders, as one climbs up the income ladder.

There is another way of interpreting the Canberra index, which involves invoking the notion of income-related relative deprivation (see Chakravarty, Chattopadhyay and Majumder 1995; Chakravarty 2007). Specifically, consider an income-recipient with income x, in a situation where the highest income in the distribution is $\bar{x}$. Let $D(x)$ be an indicator of the distribution-relative deprivation status of the person with income x. It seems reasonable to require that $D(x)$ should decline with x and—if we are interested in equity-sensitivity—that $D(x)$ should decline with x at an increasing rate. Further, in terms of a simple normalization requirement which ensures that the deprivation status of an individual is encompassed in the interval $[0,1]$, we can demand that $D(0) = 1$ and $D(\bar{x}) = 0$. Briefly, a reasonable perspective on how deprivation status might be expected to change with income would require the $D(x)$ graph to be a declining and strictly convex curve over the range $[0,1]$, with $D(0)=1$ and $D(\bar{x})=0$, as featured in Figure 1. A specific deprivation function that satisfies these properties is what one may call the Canberra deprivation function, given by: $D^C(x)=[\mu-\mu(x)]/[\mu+\mu(x)]$ for all $x \in [0,\bar{x}]$, where μ is the mean of the distribution under study, and $\mu(x)$ is the mean income of all recipients with incomes not exceeding x. Thus, given any ordered income *n*-vector $\mathbf{x}=(x_1,...,x_i,...,x_n)$, an inequality measure associated with the vector can be written as a simple average of all the individuals' Canberra deprivation functions: $I(\mathbf{x})=(1/n(\mathbf{x}))\sum_{i=1}^{n(\mathbf{x})} D^C(x_i)$. I is, precisely, the Canberra inequality measure C.

Figure 1: The Canberra deprivation function



Source: author's illustration.

The use of individual deprivation functions is reminiscent of one widely employed method of constructing poverty measures. It is customary, in the construction of poverty indices, to specify a poverty line z as a level of income below which a person is considered impoverished; and given any $\mathbf{x} \in \mathbf{X}$, if $Q(\mathbf{x}) \equiv \{i \in N(\mathbf{x}) | x_i < z\}$ and $q(\mathbf{x}) \equiv \# Q(\mathbf{x})$, then the deprivation status of an individual i, if i is poor, is taken to be some increasing function of the shortfall of that person's income from the poverty line $(i=1,...,q(\mathbf{x}))$, and zero if the person is non-poor $(i=q(\mathbf{x})+1,...,n(\mathbf{x}))$. A measure of poverty can then be taken to be a simple average of all individuals' deprivation functions. Unsurprisingly, under certain well-defined limits, poverty measures are transformed into inequality measures: thus, and as is well known, when the poverty line z is replaced by the mean income μ of a distribution, and when q is replaced by n, the Sen index of poverty becomes the Gini coefficient of inequality; the Watts index of poverty becomes one of Theil's inequality measures; and one member of the Foster-Greer-Thorbecke family of poverty measures becomes the squared coefficient of variation. The Bonferroni deprivation function has also been employed in the derivation of a poverty index, as has been demonstrated in Giorgi and Crescenzi (2001).

Indeed, in a distance function approach to the construction of a poverty index, Subramanian (2009) suggests the following procedure. Given any ordered income *n*-

vector $\mathbf{x} = (x_1,...,x_q, x_{q+1},...,x_n) \in \mathbf{X}$, one can define the associated n-vectors $\mathbf{c_x} = (x_1,...,x_q, z,...,z)$ (which is what Takayama 1979 calls a 'censored' vector), $\mathbf{z_x} = (z,...,z)$ (which is just a vector, of the same dimensionality as **x**, with the smallest mean that is compatible with a complete absence of poverty), and $\mathbf{0_x} = (0,...,0)$ (which is the vector representing maximal poverty, with every person receiving zero income). A normalized index of poverty $P(\mathbf{x}; z)$ can now be derived as the ratio of two vector distances: the distance between the vectors $\mathbf{c_x}$ and $\mathbf{z_x}$ (which is the gap between the 'actual poverty situation' and the 'no poverty situation') and the distance between the vectors $\mathbf{0_x}$ and $\mathbf{z_x}$ (which is the gap between the 'complete poverty situation' and the 'no poverty situation'): $P(\mathbf{x}; z) = \delta(\mathbf{z_x}, \mathbf{c_x}) / \delta(\mathbf{z_x}, \mathbf{0_x})$. Equally, one may write a poverty index as: $P(\mathbf{x}; z) = \delta(\mathbf{z_x}, \hat{\boldsymbol{\mu}}_\mathbf{x}^\mathbf{c}) / \delta(\mathbf{z_x}, \mathbf{0_x})$, where $\hat{\boldsymbol{\mu}}_\mathbf{x}^\mathbf{c}$ is derived from $\mathbf{c_x}$ as the n-vector $(\mu_1,...,\mu_q, z,...,z)$. If the distance function employed is the Canberra distance function δ_C, then we obtain the poverty index $P^C(\mathbf{x}; z) = (1/n(\mathbf{x})) \sum_{i=1}^{q(\mathbf{x})} \left[\frac{z - \mu_i(\mathbf{x})}{z + \mu_i(\mathbf{x})} \right]$. The superscript C on P stands for 'Canberra', and the 'Canberra' poverty measure is analogous to the 'Bonferroni' poverty measure derived by Giorgi and Crescenzi (2001). When z is replaced by $\mu(\mathbf{x})$ and $q(\mathbf{x})$ by $n(\mathbf{x})$ in the expression for P^C, we just recover—as we might expect—the Canberra inequality measure C.

When all incomes are equal, the value of the Canberra inequality measure is zero; and when a single person appropriates the entire income, the measure assumes a value of $(n-1)/n$. It is easy to verify that the index satisfies the commonly advanced properties of transfer, symmetry, continuity, and scale and replication invariance. Of specific interest are the properties of sub-group decomposability and transfer-sensitivity, the first of which is violated by C, and the second is satisfied. This is elaborated on in the following two sections.

5 Sub-group decomposability

The measure C cannot be expressed as an exact sum of a 'within-group' component and a 'between-group' component of inequality. The amenability of a poverty measure to such an additive split is what is commonly referred to as the property of sub-group decomposability. Indeed, the Canberra measure does not even satisfy the weaker property of sub-group consistency (see Shorrocks 1988), which requires that, other things equal, an increase in any one sub-group's inequality should raise overall inequality. The following simple numerical example demonstrates this. Consider the two ordered 4-vectors of income $\mathbf{x} = (2,4,5,5)$ and $\mathbf{y} = (2,5,5,5)$. Assume that there are two groups 1 and 2, and that the sub-group vectors of income can be written as: $\mathbf{x}^1 = (2,4)$, $\mathbf{y}^1 = (2,5)$, and $\mathbf{x}^2 = \mathbf{y}^2 = (5,5)$. Employing the expression for C provided in equation (2), it can be verified that $C(\mathbf{x}^1) = 0.1$, $C(\mathbf{y}^1) = 0.136$, and $C(\mathbf{x}^2) = C(\mathbf{y}^2) = 0$: since sub-group 1's inequality has gone up in **y** vis-à-vis **x**, with sub-group 2's inequality level remaining unchanged, one should expect—by sub-group consistency—that $C(\mathbf{y}) > C(\mathbf{x})$. However, and as can be easily checked, it turns out that $C(\mathbf{y})(=0.122) < C(\mathbf{x})(=0.130)$: sub-group consistency is violated by C.

C violates sub-group consistency for the same reason that the Gini and Bonferroni indices violate the axiom, namely, via a violation of the property of 'independence of irrelevant alternatives', and—through that route—the property of 'contraction consistency' (see Sen 1973). Specifically, and as the numerical example we have employed demonstrates, when we focus on any particular sub-group, the incomes outside of that sub-group ought—if one sets store by sub-group consistency—to become 'irrelevant' for an assessment of the sub-group's inequality level; however, these incomes do become material in the case of the Canberra measure because as a result of the contraction from the whole group to a sub-group, both the group-specific means (the μ's) and the 'truncated means' (the μ_i's) (could) change from what they were before the contraction. These changes amount to a violation of 'the independence of irrelevant alternatives', with a consequential violation of contraction consistency. The normative question however remains as to whether the incomes outside of the sub-group under consideration ought really to be treated as irrelevant alternatives such that the sub-group level of inequality must be seen to be independent of them (see Foster and Sen 1997). An implication of this is the following. Suppose the population is partitioned into K mutually exclusive and completely exhaustive sub-groups; that μ^j is the mean income of sub-group *j* and μ_i^j the mean income of the *i*th poorest person in sub-group *j*, whose members constitute the set N^j of individuals $(j = 1,...,K)$. For every $j \in \{1,...,K\}$, let $\lambda^j : N^j \to N$ be a one-to-one mapping such that the *i*th poorest person in N^j is the same as the $\lambda^j(i)$th poorest person in N, for every $i \in N^j$. 'Properly' speaking, the Canberra inequality measure for the *j*th group ought to be written as:

$$(6)\ C^j = (1/n^j)\sum_{i\in N^j}\left[\frac{\mu^j - \mu_i^j}{\mu^j + \mu_i^j}\right].$$

If, however, one believes that the relation of elements within each sub-group to the entire community *is* relevant to an assessment of sub-group inequality, then it would be legitimate to write the Canberra measure for sub-group *j* as in equation (7) below:

$$(7)\ \hat{C}^j = (1/n^j)\sum_{i\in N^j}\left[\frac{\mu - \mu_{\lambda^j(i)}}{\mu + \mu_{\lambda^j(i)}}\right].$$

Using equation (7), one can write the Canberra measure as an index that is decomposable after one fashion:

$$(8)\ C = \sum_{j=1}^{K}(n^j/n)\hat{C}^j\ .$$

Equation (8) reflects the sort of sub-group decomposition which Podder (1993) performs for the Gini coefficient of inequality, wherein sub-group income-ranks of individuals are replaced by the income-ranks of these same individuals in the overall vector of incomes. 'Decomposability', as in equation (8), is now interpreted in the same way in which the decomposability of a poverty measure is conventionally understood, namely as the ability to write the measure as a population share-weighted sum of sub-group poverty levels. (This enables one to identify the contribution of any particular sub-group to overall inequality: equation (8) suggests that the proportionate contribution of sub-group j to aggregate inequality is $((n^j/n)\hat{C}^j)/C$). This is not, of course, decomposability in the Theil (1967) sense of an exact additive sum of a 'within-group' and a 'between-group' component of inequality—but then, as Podder (1993: 263) sensibly observes: '… without questioning its usefulness, it can be contended that the Theil type of decomposition is not the only type of decomposition one can think of. If the general purpose is to get an idea of the contribution of each of the groups to total inequality … it is possible to think of other types of decomposition.'

6 Transfer-sensitivity

Transfer-sensitivity is the requirement that an inequality measure should be more sensitive to income transfers at the lower than at the upper end of an income distribution. There are alternative ways of giving expression to this property which Kolm (1976) called the 'principle of diminishing transfers'. Under one version, a given income transfer between two individuals a fixed *number of persons* apart should have a greater effect on inequality the poorer the pair of persons involved in the transfer is; under another version, a given income transfer between two individuals a fixed *income* apart should have a greater effect on inequality the poorer the pair of persons involved in the transfer is (see Foster 1985). For our purposes, we shall define transfer-sensitivity under the constraint that the pair of individuals involved in the transfer are both a fixed population *and* a fixed income apart. The following definitions are in order.

Given an ordered income *n*-vector $\mathbf{x} = (x_1,...,x_j,...,x_k,...,x_n)$, a *progressive rank-preserving transfer* of income between two persons k and j is one in which $x_k > x_j$ and a transfer of $\Delta \le (x_k - x_j)/2$ takes place from k to j.

The Transfer Axiom requires that for any two ordered income *n*-vectors $\mathbf{x},\mathbf{y} \in \mathbf{X}$, if $\mathbf{y}$ has been derived from $\mathbf{x}$ through a progressive rank-preserving transfer of income from some person k to some person j, i.e. $y_i = x_i \forall i \notin \{j,k\}$ for some $j,k \in N(\mathbf{x})$ satisfying $y_j = x_j + \Delta, y_k = x_k - \Delta, x_k > x_j$ and $\Delta \le (x_k - x_j)/2$, then $I(\mathbf{y}) < I(\mathbf{x})$.

Transfer-Sensitivity, as we define it, requires the following: For all ordered income *n*-vectors $\mathbf{x},\mathbf{y},\mathbf{w} \in \mathbf{X}$, if $\mathbf{y}$ is derived from $\mathbf{x}$ through a progressive rank-preserving transfer of income from some k to some j, and $\mathbf{w}$ is derived from $\mathbf{x}$ through a progressive rank-preserving transfer of income from some q to some p, with $k - j = q - p \equiv t > 0$, and $x_k - x_j = x_q - x_p \equiv \Delta > 0$, then $I(\mathbf{x}) - I(\mathbf{y}) > I(\mathbf{x}) - I(\mathbf{w}) > 0$.

It is well known that the Gini coefficient violates transfer-sensitivity: so long as the area enclosed by the Lorenz cure and the diagonal of the unit square in which the curve is drawn is the same for any intersecting pair of Lorenz curves, the value of the Gini coefficient will be the same for both distributions, irrespective of whether the Lorenz curve bulges at the top or at the bottom of the distribution. The Bonferroni index, however, is transfer-sensitive (see Chakravarty 2007). So is the Canberra measure.

To see this, imagine that the antecedents in the statement of the Transfer-Sensitivity Axiom have been satisfied for some triple of *n*-vectors of income $\mathbf{x},\mathbf{y},\mathbf{w} \in \mathbf{X}$.Then, it can be verified that, if μ is the common mean shared by the three distributions,

(9) $$C(\mathbf{x}) - C(\mathbf{y}) = (2\mu\Delta/n)\left[\frac{1}{(\mu+\mu_j)\{j(\mu+\mu_j)+\Delta\}} + \ldots + \frac{1}{(\mu+\mu_{j+t-1})\{(j+t-1)(\mu+\mu_{j+t-1})+\Delta\}}\right]$$

and

(10) $$C(\mathbf{x}) - C(\mathbf{w}) = (2\mu\Delta/n)\left[\frac{1}{(\mu+\mu_p)\{p(\mu+\mu_p)+\Delta\}} + \ldots + \frac{1}{(\mu+\mu_{p+t-1})\{(p+t-1)(\mu+\mu_{p+t-1})+\Delta\}}\right]$$

Since the right hand sides of equations (9) and (10) are positive, the Transfer axiom is verified. Further, and since $j < p$ and the truncated means are arranged in non-descending order, it follows that $1/(\mu+\mu_{j+i})\{(j+i)(\mu+\mu_{j+i})+\Delta\}$ > $1/(\mu+\mu_{p+i})\{(p+i)(\mu+\mu_{p+i})+\Delta\}$ $\forall i \in \{0,1,\ldots,t-1\}$, which is sufficient to verify that C satisfies transfer-sensitivity.

If transfer-sensitivity is regarded as an appealing property, then the Bonferroni and Canberra measures score over the Gini coefficient in this respect.

7 The Canberra curve

Given any ordered *n*-vector of incomes $\mathbf{x} \in \mathbf{X}$, we know that the Lorenz curve is defined by the relationship

(11) $$L(\mathbf{x}; i/n) = (1/n\mu)\sum_{j=1}^{i} x_j \ .$$

Similarly, we can define the Canberra curve in terms of the following relationship:

(12) $$R(\mathbf{x}; i/n) = \frac{(i/n) - (\sum_{j=1}^{i} x_j / n\mu)}{(i/n) + (\sum_{j=1}^{i} x_j / n\mu)}\left(= \frac{\mu - \mu_i}{\mu + \mu_i}\right).$$

If we designate i/n by P_i and $(1/n\mu)\sum_{j=1}^{i} x_j$ by L_i, then L_i is the cumulative income share of the poorest P_ith fraction of the population, and the plot of L_i against P_i is just the Lorenz curve. In like fashion, if we designate $\frac{(i/n) - (\sum_{j=1}^{i} x_j / n\mu)}{(i/n) + (\sum_{j=1}^{i} x_j / n\mu)}$ by R_i, then we can see from (12) that the Canberra curve is obtained by plotting the points $R_i \equiv (P_i - L_i)/(P_i + L_i)$ against the points P_i $(i = 1,\ldots,n)$. The curve can be drawn within the unit square as a non-increasing graph from $(0,1)$ to $(1,0)$ of the square. For an illustrative numerical example, consider the ordered 5-vector $\mathbf{x} = (10,15,20,24,41)$ whose mean μ is 22. The co-ordinates of the Canberra curve for the distribution $\mathbf{x}$ can be derived as in the last two columns of the following table:

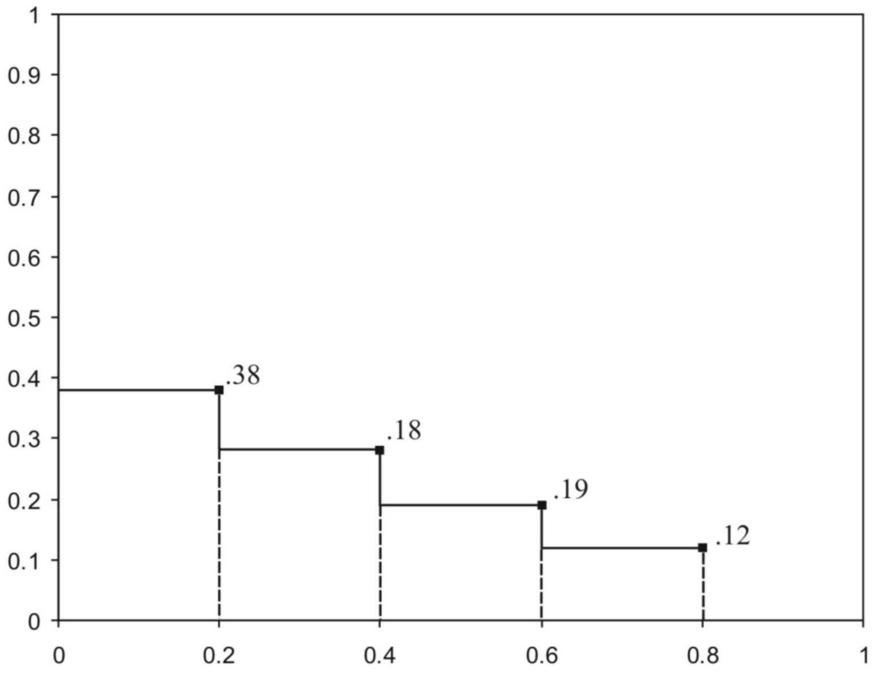
Co-ordinates of the Canberra curve for the vector $\mathbf{x} = (10,15,20,24,41)$

i	x_i	μ_i	$\mu - \mu_i$	$\mu + \mu_i$	$P_i \equiv i/n$	$R_i \equiv \dfrac{\mu - \mu_i}{\mu + \mu_i}$
1	10	10	12	32	0.2	0.38
2	15	12.5	9.5	34.5	0.4	0.28
3	20	15	7	37	0.6	0.19
4	24	17.25	4.75	39.25	0.8	0.12
5	41	22	0	44	1.0	0

A plot of the R_i against the P_i presented in the last two columns of the table above yields the Canberra curve as a step function—see Figure 2. It is easy to see that the area beneath the Canberra curve is just the value of the Canberra measure of inequality. One can also see that as n becomes large, the Canberra curve can be approximated by a continuous curve obtained by connecting the plotted points of the curve with 'piece-wise' linear segments. Two examples of possible Canberra curves are presented in Figure 3. One curve lies everywhere below what Kakwani (1980) calls the 'alternative diagonal' drawn from $(0,1)$ to $(1,0)$ of the unit square; the other curve lies everywhere above this diagonal.[1] A relationship analogous to that of Lorenz dominance can be defined for the Canberra curve: for any two distributions $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}$ will be said to Canberra-dominate $\mathbf{y}$, written $\mathbf{x} \succ_C \mathbf{y}$, if and only if the Canberra curve for $\mathbf{x}$ lies somewhere below and nowhere above the Canberra curve for $\mathbf{y}$. For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $\mathbf{x} \succ_C \mathbf{y}$, then one can say that $\mathbf{x}$ displays unambiguously less inequality than $\mathbf{y}$. In Figure 3, the strictly convex Canberra curve obviously dominates the strictly concave curve.

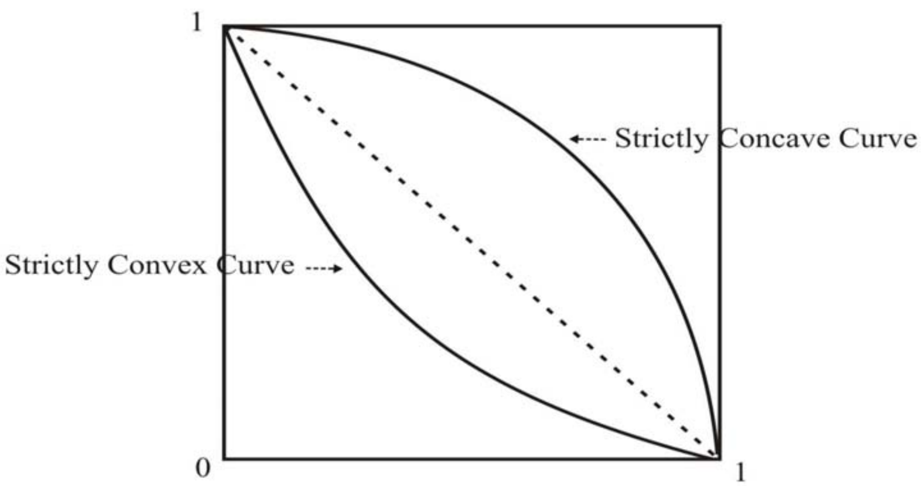
[1] It is straightforward that if the Canberra curve is uniformly convex, then it will lie everywhere below the alternative diagonal, while if it is uniformly concave, it will lie everywhere above the alternative diagonal. The area below the alternative diagonal is 0.5, which serves as a sort of benchmark: if the Canberra curve is uniformly convex, then the value of the Canberra measure is less than 0.5, and the other way around if the curve is uniformly concave.

Figure 2: A step-function Canberra curve drawn for the vector (10, 15, 20, 24, 41)



Source: author’s illustration.

Figure 3: Two examples of possible Canberra curves



Source: author’s illustration.

Finally, visual confirmation of the transfer-sensitivity of the Canberra measure is available from the following elementary numerical example. Suppose $\mathbf{a} = (20,30,40,50)$ and that $\mathbf{x}$ and $\mathbf{y}$ are derived from $\mathbf{a}$ through, respectively, a transfer of 5 units of income from person 2 to person 1, and a transfer of 5 units from person 4 to person 3, so that $\mathbf{x} = (25,25,40,50)$, $\mathbf{y} = (20,30,45,45)$, and $\mathbf{x}$ and $\mathbf{y}$ share the same mean $\mu = 35$.

The co-ordinates of both the Lorenz and the Canberra curves are derived in the tables following:

Co-ordinates of the Lorenz and the Canberra curves for the vector $\mathbf{x} = (20,25,40,50)$

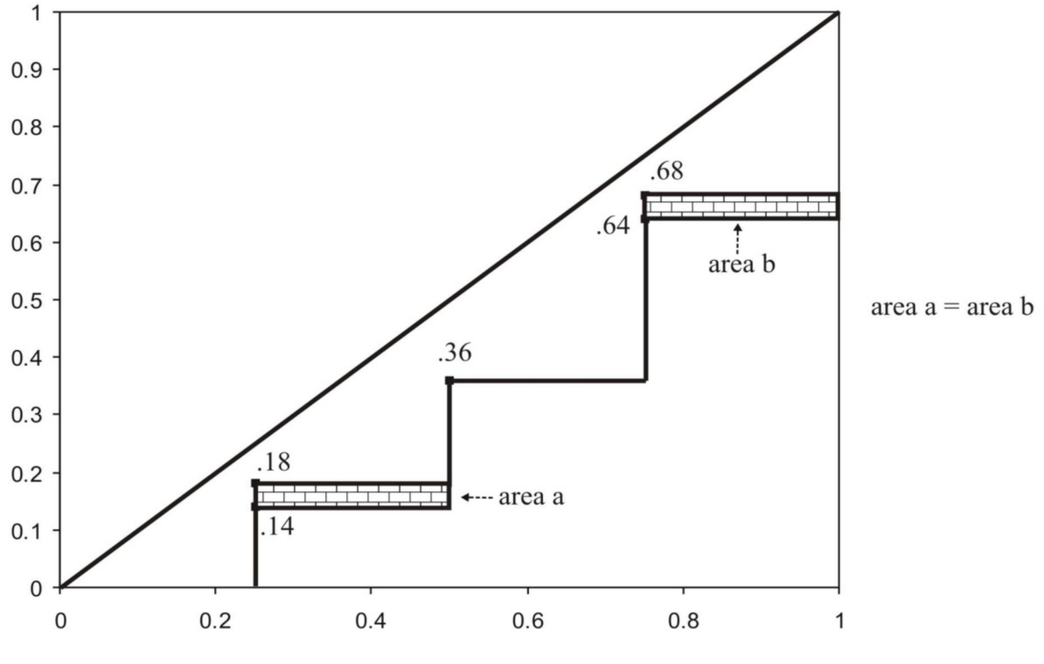
i	x_i	μ_i	$\mu - \mu_i$	$\mu + \mu_i$	$P_i \equiv i/n$	$R_i \equiv \frac{\mu - \mu_i}{\mu + \mu_i}$	$L_i \equiv (1/n\mu)\sum_{j=1}^{i} x_j$
1	25	25	10	60	0.25	0.17	0.18
2	25	25	10	60	0.50	0.17	0.36
3	40	30	5	65	0.75	0.08	0.64
4	50	35	0	70	1.00	0	1.00

Co-ordinates of the Lorenz and the Canberra curves for the vector $\mathbf{y} = (20,30,45,45)$

i	x_i	μ_i	$\mu - \mu_i$	$\mu + \mu_i$	$P_i \equiv i/n$	$R_i \equiv \frac{\mu - \mu_i}{\mu + \mu_i}$	$L_i \equiv (1/n\mu)\sum_{j=1}^{i} x_j$
1	20	20	15	55	0.25	0.27	0.14
2	30	255	10	60	0.50	0.17	0.36
3	45	31.67	3.33	66.67	0.75	0.05	0.68
4	45	355	0.75	70	1.00	0	1.00

From the step functions representing the Lorenz and the Canberra curves for the two distributions, one can see that the Gini coefficients are the same for both $\mathbf{x}$ and $\mathbf{y}$ (the area a in Figure 4a is the same as the area b), while the Canberra measure is larger for $\mathbf{y}$ than for $\mathbf{x}$ (the area a in Figure 4b is larger than the area b).

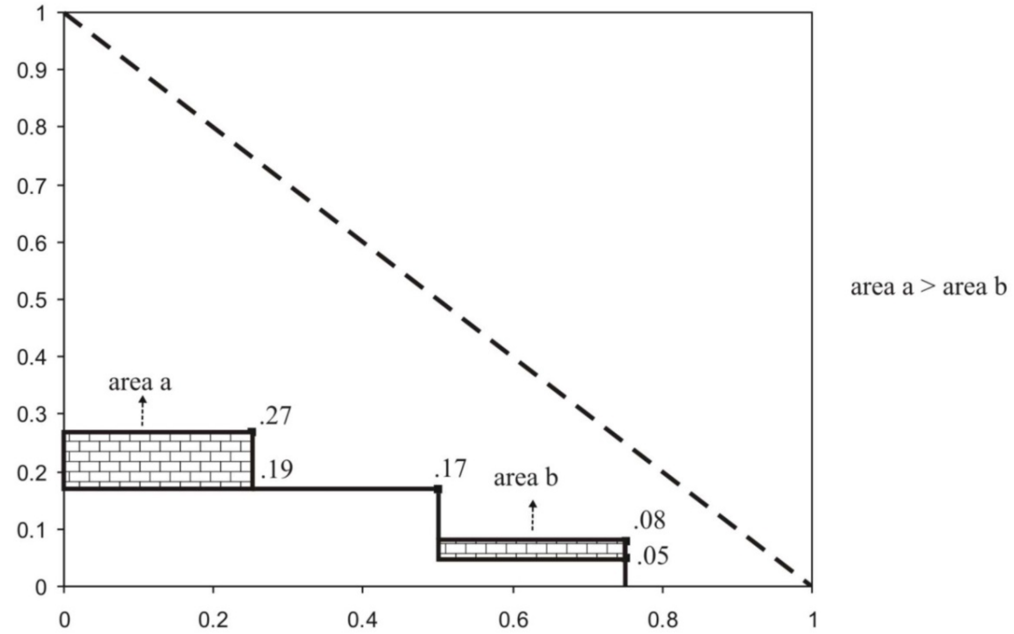
Figure 4a: Step-function Lorenz curves for the vectors x = (25, 25, 40, 50) and y = (20, 30, 45, 45)



Note: the Lorenz curve for *y* first lies below and then above the lorenz curve for *x*.

Source: author's illustration.

Figure 4b: Step-function Canberra curves for the vectors x = (25, 25, 40, 50) and y = (20, 30, 45, 45)



Note: the Canberra curve for *y* first lies above and then below the Canberra Curve for *x*.
Source: author's illustration.

8 Concluding observations

This paper has been a variation on the theme of the Bonferroni inequality index, which has been subjected to rigorous analysis by, among others, Barcena and Imedio (2000), Giorgi and Crescenzi (2001), Chakravarty (2007), and Imedio-Olmedo et al. (2011). The possible novelty of the paper resides in the use of a distance function—the Canberra distance function, as it happens—as a natural approach to take in the measurement of both inequality and poverty. An application of the Canberra distance function to an assessment of inequality leads to a measure of disparity—here called the 'Canberra measure'—which turns out to be closely related to the Bonferroni index, and also to the Gini coefficient of inequality. A curve analogous to the Lorenz curve, and referred to as the Canberra curve in the paper, is derived and discussed.

Also discussed are some properties of the Canberra inequality measure, with specific reference to the features of decomposability and transfer-sensitivity. The principal merit of the Canberra measure vis-à-vis the Gini coefficient is that, unlike the latter, it satisfies the property of transfer-sensitivity. The emphasis of the paper has been mainly on a simple and systematic derivation and presentation of an inequality measure with a known ancestry in two other distinguished measures. The paper is thus best viewed as an effort in consolidation and, it is hoped, useful exposition.

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